we can write the value of ω in the form

$$\omega = 2\pi - \left[\arctan \frac{(x-a)\Delta_1}{(a-y)(a+z)} + \arctan \frac{(x-a)\Delta_2}{(a+y)(a+z)} + \arctan \frac{(x-a)\Delta_3}{(a+y)(a-z)} + \arctan \frac{(x-a)\Delta_7}{(a-y)(a-z)} \right]. \tag{3}$$

Case B. $\mathbf{x > a,\ y > a,\ -a < z < a}$. In this case two faces are visible and $\omega = \omega_2 + \omega_3$. If we combine the terms that contain the same radical by means of the formula

$$\arctan(m) + \arctan(n) = \arctan \frac{m+n}{1-mn}$$

we obtain, after further reduction,

$$\omega = 2\pi - \left[\arctan \frac{(a+z)\Delta_1}{(x-a)(y-a)} + \arctan \frac{(x-a)\Delta_2}{(y+a)(a+z)} + \arctan \frac{(x-a)\Delta_3}{(y+a)(a-z)} + \arctan \frac{(a-z)\Delta_7}{(x-a)(y-a)} + \arctan \frac{(y-a)\Delta_5}{(a-z)(x+a)} + \arctan \frac{(y-a)\Delta_6}{(x+a)(a+z)} \right]. \tag{4}$$

Case C. $\mathbf{x > a,\ y > a,\ z > a}$. In this case three faces are visible and $\omega = \omega_1 + \omega_2 + \omega_3$. After combining terms as before we obtain

$$\omega = 2\pi - \left[\arctan \frac{(z+a)\Delta_1}{(x-a)(y-a)} + \arctan \frac{(x+a)\Delta_5}{(y-a)(z-a)} + \arctan \frac{(y+a)\Delta_3}{(z-a)(x-a)} + \arctan \frac{(z-a)\Delta_4}{(x+a)(y+a)} + \arctan \frac{(x-a)\Delta_2}{(y+a)(z+a)} + \arctan \frac{(y-a)\Delta_6}{(z+a)(x+a)} \right]. \tag{5}$$

It is not necessary to consider any more than these three cases since, for any given value of ω, the surfaces given by equations (3), (4), and (5) will be symmetrical with respect to the coördinate planes and the origin. Hence it is sufficient to study the formulas for one octant with different regions according to the number of faces visible. If the upper face alone is visible we can obtain the value of the solid angle by simply interchanging x and z in equation (3); if faces $P_1P_2P_3P_7$ and $P_2P_3P_4P_8$ are visible, simply change y to $-y$ in (4), etc.

It may be interesting to note that from any point of sight the cube will project into either a quadrilateral or a hexagon and in each of the equations (3), (4), (5) the radicals, which we have denoted by Δ, represent the distances from the point of sight to the vertices of that quadrilateral or hexagon.

3. The solid angle subtended by the cube may also be obtained from the relation $\omega = 2\pi - \sigma$ where σ is the length of that curve on the unit sphere which is polar to the intersection of the cone and the unit sphere. In this problem σ is the sum of the dihedral angles of the cone. Of course this cone is really a pyramid since its directrix is made up of straight lines.

If in each of the three cases we write the equations of the planes forming the lateral faces of the pyramids and then find the value of σ for each pyramid, the formula $\omega = 2\pi - \sigma$ leads to equations (3), (4), (5).

4. It may be interesting to note the behavior of these formulas as the point P moves away from the cube in any direction. If we change to polar coördinates and then express ω_1 as a power series in $1/r$, it will be found that the series begins with the term in $1/r^2$. In fact

$$\omega_1 = 4a^2 \cos(n, r) \times 1/r^2 + \cdots \text{ higher powers of } 1/r,$$

where (n, r) is the angle that the radius vector makes with the normal to the face. It will be found that ω_2 and ω_3 may be written in the same form. Hence we see that in each of the three cases (when one, two or three faces are visible) the apparent size of the cube will approach zero as the point of sight approaches infinity in any direction; and also that ωr^2 will approach

$$\Sigma(\text{area of face}) \times (\text{cosine of the angle between the radius vector and the normal to the face}).$$

In other words, as the point of sight approaches infinity in any direction, each of the faces of the cube acts more and more as if it were a single differential element $\cos\theta\, dS/r^2$.

5. If we combine certain terms in each of the equations (3), (4), (5) we are led to the following results: (*a*) when only one face is visible the solid angle may have any value between zero and 2π, being equal to 2π only when the point of sight lies on the face of the cube; (*b*) when only two faces are visible the solid angle lies between zero and π, being equal to π only when the point of sight lies on one edge of the cube; (*c*) when three faces are visible the solid angle lies between zero and $\pi/2$, being equal to $\pi/2$ only when the point of sight is at one corner of the cube.

If we let ω be constant in equations (3), (4), (5) we have in each case the equation of a surface. The apparent size of the cube will not vary as long as the point of sight remains on this surface. We propose now to study the forms of these surfaces for different values of ω.

6. Let us assume all space outside the cube to be divided into the following regions: (*a*) six prisms bounded by the six faces of the cube produced and extending, one from each face of the cube, to infinity; (*b*) twelve wedge-shaped regions extending, one from each edge, to infinity and bounded by the two faces which are perpendicular to this edge and the two faces which pass through the edge; (*c*) eight three-edged regions extending, one from each corner of the cube, to infinity and bounded by the three faces produced which meet in the given corner.

Then if ω has a constant value, such that all three of our surfaces exist, we have:

Case A. six distinct surfaces, one in each of the regions (a), symmetrical in pairs with respect to the three lines joining the center of the cube to the mid-points of the faces and symmetrical in pairs with respect to the origin;

Case B. twelve distinct surfaces, one in each of the regions (b), symmetrical in pairs with respect to the lines joining the mid-points of opposite edges of the cube and symmetrical in pairs with respect to the origin;

Case C. eight distinct surfaces, one in each of the regions (c), symmetrical in pairs with respect to the diagonals of the cube and symmetrical in pairs with respect to the origin.

It is not necessary to study each of these twenty-six different surfaces since any one of them may be obtained from one of the equations (3), (4), (5) by various permutations of letters and signs. For simplicity we shall refer to the surface from which only one face is visible as surface I or $F_1(x, y, z) = 0$; the surface from which two faces are visible we shall call surface II or $F_2(x, y, z) = 0$; and the surface from which three faces are visible we shall call surface III or $F_3(x, y, z) = 0$. Their equations are given by (3), (4), (5).

7. We propose now to obtain the equations of the curves in which the planes of the faces produced intersect surfaces I, II, and III. Of course it must be understood in all that follows that $\omega < \pi/2$, otherwise some of the surfaces may not exist. The plane $y = a$ separates the region of space in which surface I lies from that in which surface II is found. It can be shown easily that both these surfaces are cut by the plane $y = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan \frac{(x-a)\Delta_2}{2a(a+z)} + \arctan \frac{(x-a)\Delta_3}{2a(a-z)} \right]. \tag{6}$$

Likewise surfaces II and III are both cut by the plane $z = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan \frac{2a\Delta_1}{(x-a)(y-a)} + \arctan \frac{(x-a)\Delta_2}{2a(y+a)} + \arctan \frac{(y-a)\Delta_6}{2a(x+a)} \right]. \tag{7}$$

Hence we see that although we have twenty-six different surfaces defined by twenty-six distinct equations, they come together in pairs on the faces of the cube produced and form a single surface completely surrounding the cube and continuous everywhere.

8. Let us see whether any pair of these surfaces have a common tangent plane at any point of their curve of intersection. The direction cosines of the normal to the tangent plane to any surface are proportional to $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, where $F(x, y, z) = 0$ is the equation of the surface. These partial derivatives may be easily obtained from equations (3), (4), (5), and it is not necessary to write their values here.

Since surfaces I and II meet on the plane $y = a$ we must investigate the values

of $\partial F_1/\partial x$, $\partial F_1/\partial y$, $\partial F_1/\partial z$, $\partial F_2/\partial x$, $\partial F_2/\partial y$, $\partial F_2/\partial z$ for $y = a$. If at any point on the curve of intersection we find the first three of these derivatives respectively equal to or proportional to the last three then the two surfaces have a common tangent plane at that point. It can be easily shown that $\partial F_1/\partial x \equiv \partial F_2/\partial x$ and $\partial F_1/\partial z \equiv \partial F_2/\partial z$, and $\partial F_1/\partial y \not\equiv \partial F_2/\partial y$. Hence the two surfaces do not have a common tangent plane at *every* point of their curve of intersection. Let us see whether they have a common tangent plane at *any* point of their curve of intersection. If $\partial F_1/\partial y = \partial F_2/\partial y$ it easily follows that

$$(x + a)(a - z)\Delta_1 - (x - a)(a - z)\Delta_6 = (x - a)(a + z)\Delta_5 - (x + a)(a + z)\Delta_7.$$

After a rather lengthy reduction it can be shown that this equation is not possible unless $x = a$ and $z = \pm a$, i. e., unless the curve of intersection passes through one corner of the cube. This cannot happen except for $\omega = \pi/2$, which will be considered later. Hence, if $\omega < \pi/2$, the surfaces I and II have no common tangent plane at any point of their curve of intersection.

Likewise it can be shown that, if $\omega < \pi/2$, surfaces II and III have no common tangent plane at any point of their curve of intersection.

9. We have said that each of the surfaces I, II, and III is symmetrical with respect to a certain line which we shall call the axis of the surface. The X-axis is the axis of surface I; the line joining the origin to the point $(a, a, 0)$ is the axis of surface II; and surface III has for its axis the line joining the origin to the point (a, a, a), the corner of the cube. We propose to show that each axis is a normal to its surface.

The axis of surface I will cut it at some point on the X-axis. Hence we must investigate the values of $\partial F_1/dx$, $\partial F_1/\partial y$, $\partial F_1/\partial z$ when $y = z = 0$. We find

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_7, \quad \text{and} \quad \frac{\partial F_1}{\partial x} = \frac{-8a^2}{\Delta_1[(x-a)^2 + a^2]}, \quad \frac{\partial F_1}{\partial y} = 0, \quad \frac{\partial F_1}{\partial z} = 0.$$

Hence the direction cosines of the normal at the point $(x, 0, 0)$ are $1, 0, 0$. In other words the X-axis is the normal.

The axis of surface II will cut the surface at some point on the line $y = x$, $z = 0$. If we put $y = x$ and $z = 0$ we find $\Delta_1 = \Delta_7$, $\Delta_2 = \Delta_3 = \Delta_5 = \Delta_6$, and $\partial F_2/\partial x = \partial F_2/\partial y \neq 0$, and $\partial F_2/\partial z = 0$. Hence the direction cosines of the normal at this point are proportional to 1, 1, 0; i. e. $\lambda = 1/\sqrt{2}$, $\mu = 1/\sqrt{2}$, $\nu = 0$. Hence the axis is the normal.

The axis of surface III will cut the surface at some point where $z = y = x$. If we put $z = y = x$ we find $\Delta_1 = \Delta_3 = \Delta_5$ and $\Delta_2 = \Delta_4 = \Delta_6$. Also $\partial F_3/\partial x = \partial F_3/\partial y = \partial F_3/\partial z$. Hence the direction cosines of the normal are $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$. In other words the line $y = z = x$ is the normal.

10. In order to make the surfaces more vivid geometrically we shall find the coördinates of several points on them. Let us first find the length of the intercept of each surface on its axis.

Surface I. Put $z = y = 0$ in equation (3) and obtain

$$\frac{\pi}{2} - \frac{\omega}{4} = \arctan \frac{\sqrt{(x-a)^2 + 2a^2}}{a^2}.$$

Solving for x we have $x = a[1 \pm \sqrt{\csc(\omega/4) - 1}]$. We must take the positive sign in order to be on the positive side of the face. Hence

$$x = a\left[1 + \sqrt{\csc\frac{\omega}{4} - 1}\right]. \tag{8}$$

This gives us the length of $OA_z = OA_y = OA_x$. (Fig. 2.)

Surface II. Putting $z = 0$ and $y = x$ in equation (4), we obtain

$$\omega = 2\pi - \left[2 \arctan \frac{a\Delta_1}{(x-a)^2} + 4 \arctan \frac{(x-a)\Delta_2}{a(x+a)}\right], \quad \Delta_1^2 = 2x^2 - 4ax + 3a^2,$$

or

$$\frac{\pi}{2} - \frac{\omega}{4} = \frac{1}{2} \arctan \frac{a\Delta_1}{(x-a)^2} + \arctan \frac{(x-a)\Delta_2}{a(x+a)}, \quad \Delta_2^2 = 2x^2 + 3a^2.$$

Let

$$\varphi = \arctan \frac{a\Delta_1}{(x-a)^2}; \quad \text{then} \quad \tan\varphi = \frac{a\Delta_1}{(x-a)^2}$$

and

$$\frac{\varphi}{2} = \arctan \frac{1 - \cos\varphi}{\sin\varphi} = \arctan \frac{a}{\Delta_1}.$$

Hence

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \arctan \frac{a}{\Delta_1} + \arctan \frac{(x-a)\Delta_2}{a(x+a)},$$

then

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \cot\frac{\omega}{4} = \frac{a^2(x+a) + (x-a)\Delta_1\Delta_2}{a(x+a)\Delta_1 - a(x-a)\Delta_2}.$$

Rationalizing the denominator, we obtain

$$\cot\alpha = \frac{(x^2+a^2)\Delta_1 + (x^2-a^2)\Delta_2}{2a^2x}, \quad \text{where} \quad \alpha = \frac{\omega}{4},$$

or

$$2a^2x\cot\alpha - (x^2-a^2)\Delta_2 = (x^2+a^2)\Delta_1.$$

If we rationalize this equation we obtain

$$\begin{aligned}(1-c)^2x^8 - 4(1-c)^2ax^7 + 2(1-c)(3-5c)a^2x^6 - 4(1-c)(3-4c)a^3x^5 \\ + 3(1-c)(5-7c)a^4x^4 - 4(1-c)(3-5c)a^5x^3 + 4(3-6c+4c^2)a^6x^2 \\ - 4(1-c)(1-2c)a^7x - 2(1-c)(1+2c)a^8 = 0,\end{aligned} \tag{9}$$

where c is written in place of $\cos 2\alpha = \cos(\omega/2)$ for simplicity. This equation

has at least one positive root greater than a and this root can be found after c is known. It may happen for certain values of c that the equation has more than one positive root greater than a, in which case we must choose the root which satisfies the equation

$$\cot \frac{\omega}{4} = \frac{(x^2 + a^2)\Delta_1 + (x^2 - a^2)\Delta_2}{2a^2 x}.$$

If x_1 is the required root of equation (9) then the length of the intercept is $x_1\sqrt{2}$. This gives us the length of $OB_z = OB_y = OB_x$ in Fig. 2.

Surface III. Putting $z = y = x$ in equation (5), we obtain

$$\omega = 2\pi - \left[3 \arctan \frac{(x+a)\Delta_1}{(x-a)^2} + 3 \arctan \frac{(x-a)\Delta_2}{(x+a)^2} \right]$$

where

$$\Delta_1^2 = 2(x-a)^2 + (x+a)^2; \quad \Delta_2^2 = 2(x+a)^2 + (x-a)^2.$$

From this equation we obtain, after a rather lengthy reduction,

$$x = \pm a \sqrt{\tan\left(\frac{\pi}{6} - \frac{\omega}{12}\right) \cot \frac{\omega}{4}}. \tag{10}$$

Hence $OC = x\sqrt{3} = a\sqrt{3 \tan (\pi/6 - \omega/12) \cot (\omega/4)}$.

In the case of surfaces I and III we have found the length of the intercept as an explicit function of ω, but in the case of surface II we did not obtain a simple result. The reason for this is obvious geometrically. If we are considering only one face of the cube, surface I is symmetrical with respect to the plane of this face and hence we obtain $x - a = \pm f(\omega)$. Likewise when we are considering three faces the surface is symmetrical with respect to the origin and we obtain $x = \pm \varphi(\omega)$. This symmetry does not exist in the case of surface II. The solid angle will not be the same for two points on the axis such that $x = \pm \psi(\omega)$ if we are looking at the *same* two faces. That is, equation (9) should not have two roots of the form $x = \pm \psi(\omega)$. It can be shown easily that equations (8), (9), and (10) hold for limiting values of ω. When $\omega = 0$, each equation gives $x = \infty$; when $x = a$ we have $\omega = 2\pi$, π, and $\pi/2$ respectively.

11. We have seen that surfaces I and II are both cut by the plane $y = a$ in the curve

$$\pi - \omega = \arctan \frac{(x-a)\Delta_2}{2a(a+z)} + \arctan \frac{(x-a)\Delta_3}{2a(a-z)}. \tag{11}$$

The arc $Q_1 M_z$ (Fig. 2) represents the upper half of this curve. If we find dz/dx from (11) it can be easily shown that the curve $Q_1 M_z$ cuts the XY plane at right angles for every value of ω, and that it makes with the edge SP_7 (produced) an acute angle whose value depends upon ω. In fact this angle varies from zero to $\pi/2$ as ω varies from $\pi/2$ to zero.

Putting $z = 0$ in (11), we obtain $\pi - \omega = 2 \arctan (x-a)\Delta_2/2a^2$, whence

$$\cot (\omega/2) = \tan (\pi/2 - \omega/2) = (x-a)\Delta_2/2a^2.$$

Then

$$\cot^2(\omega/2) = (x-a)^2[(x-a)^2 + 5a^2]/4a^4$$

or

$$(x-a)^4 + 5a^2(x-a)^2 - 4a^4\cot^2(\omega/2) = 0,$$

whence

$$x - a = HM_z = a\sqrt{\frac{-5 + \sqrt{25 + 16\cot^2(\omega/2)}}{2}}. \tag{12}$$

Hence, when ω is known, we have the lengths of the lines $HM_z = HM_z' = LM_y = LM_y' = SM_x = SM_x'$.

Now putting $z = a$ in (11), we obtain after reduction

$$x - a = Q_1P_7 = 2a\sqrt{\csc\omega - 1}. \tag{13}$$

Hence we have the lengths of the lines $Q_1P_7 = Q_2P_7 = Q_3P_7$.

12. Let us now study the curve of intersection of surfaces II and III. This is the curve Q_1Q_2 (Fig. 2). Its equation is

$$\pi - \omega = \arctan\frac{2a\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{2a(y+a)} + \arctan\frac{(y-a)\Delta_6}{2a(x+a)}. \tag{14}$$

If we find the value of dy/dx from (14) we can easily show that the curve cuts the diagonal (produced) of the upper face at right angles, and that it makes with the positive direction of P_7Q_1 an obtuse angle whose size depends on the value of ω.

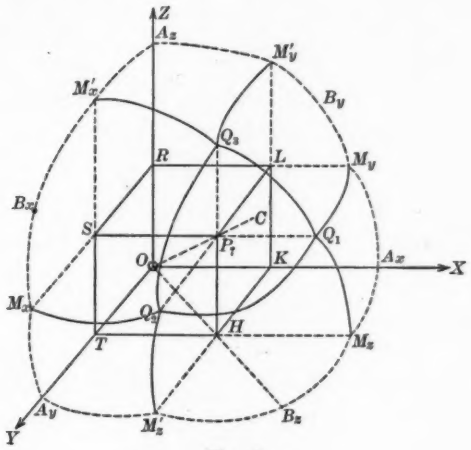


FIG. 2.

In fact as x approaches a, dy/dx approaches -1; and as x approaches ∞, dy/dx approaches $-\infty$. Hence this obtuse angle varies from $3\pi/2$ to $\pi/2$.

We can find the length of P_7Q_1 by putting $y = a$ in equation (14). This gves the same result as before.

13. Curvature properties. Let $U=\partial F/\partial x$, $V=\partial F/dy$, $W=\partial F/\partial z$, $u=\partial^2 F/\partial x^2$, $v=\partial^2 F/\partial y^2$, $w=\partial^2 F/\partial z^2$, $u'=\partial^2 F/\partial y\partial z$, $v'=\partial^2 F/\partial z\partial x$, $w'=\partial^2 F/\partial x\partial y$. Also $H=u+U(Uu'-Vv'-Ww')/VW$, $K=v+V(Vv'-Ww'-Uu')/WU$, $L=w+W(Ww'-Uu'-Vv')/UV$. Then the principal radii of curvature at any point are given by the quadratic equation

$$\frac{U^2}{H-Q/\rho}+\frac{V^2}{K-Q/\rho}+\frac{W^2}{L-Q/\rho}=0, \quad \text{where} \quad Q^2=U^2+V^2+W^2.$$

From the results already obtained it is geometrically reasonably certain that the parts of the surface between the curves of discontinuous slope are everywhere convex outward. This can be proved by showing that for any given point (x_1, y_1, z_1) the two values of ρ obtained from the equation above are both negative.

After finding the values of $\partial^2 F/\partial x^2$, $\partial^2 F/\partial y^2$, $\partial^2 F/\partial z^2$ it can be easily shown that $\partial^2 F/\partial x^2+\partial^2 F/\partial y^2+\partial^2 F/\partial z^2=0$. That is, the apparent size corresponds to a magnetic potential function.

14. We have assumed up to this point that ω is less than $\pi/2$. Let us now consider the forms of the different surfaces for other values of ω.

$\omega=\pi/2$. In this case surface III reduces to a mere point, the corner of the cube (See Fig. 3), and we have to study only surfaces I and II. The edge SP_7 is now tangent to both surfaces as we have seen before (Art. 8). Putting $\omega=\pi/2$ in (8) we find $OA_x=a(1+\sqrt{\csc(\pi/8)-1})$. The equation of the curve of intersection of surfaces I and II now becomes

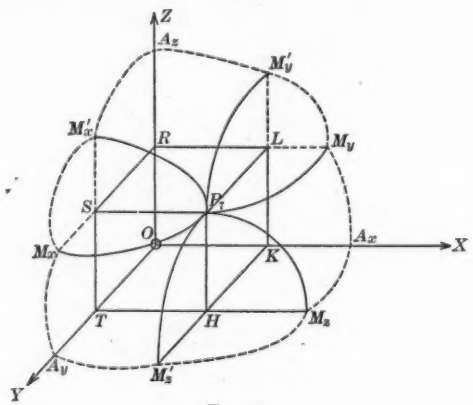


FIG. 3.

$$\frac{\pi}{2}=\arctan\frac{(x-a)\Delta_2}{2a(a+z)}+\arctan\frac{(x-a)\Delta_3}{2a(a-z)}.$$

From this equation we obtain after reduction, $4a^2(a^2-z^2)=(x-a)^2\Delta_2\Delta_3$. Squaring both sides, we obtain, after reduction,

$$(x-a)^6+2(x-a)^4(3a^2+z^2)+(x-a)^2(a^2-z^2)^2-4a^2(a^2-z^2)^2=0. \quad (15)$$

This is the algebraic equation of the curve of intersection. Putting $z = 0$ we have

$$(x-a)^6 + 6a^2(x-a)^4 + a^4(x-a)^2 - 4a^6 = 0, \quad \text{or} \quad (x-a)^4 + 5a^2(x-a)^2 - 4a^4 = 0,$$

after dividing by $(x-a)^2 + a^2$. This gives

$$HM_z = (x-a) = a\sqrt{\frac{-5+\sqrt{41}}{2}} = 0.8a,$$

approximately.

If we put $\omega = \pi/2$ in equation (12) we obtain the same result.

Putting $z = a$ in (15) we have $(x-a)^6 + 8a^2(x-a)^4 = 0$, whence $P_7Q_1 = x - a = 0$. This agrees with (13) for $\omega = \pi/2$. It can be easily shown from (15) that $dz/dx = 0$ for $z = x = a$ and $dz/dx = \infty$ for $z = 0$. Hence the point Q_1 is at P_7 and the curve P_7M_z is tangent to the edge SP_7 at P_7.

$\pi/2 < \omega < \pi$. In this case surfaces I and II still exist and OA_x is still given by (8) and HM_z by (12). The equation of the curve of intersection of the surfaces I and II, equation (11), is not satisfied by $x = a$ for any value of z, nor by $z = a$ for any value of x. In other words the curve of intersection approaches the point P_7 but never reaches it. The curve is still perpendicular to the XY plane at M_z.

$\omega = \pi$. The equation of surface II now reduces to $x = a$, $y = a$, the edge of the cube, and the curve of intersection of surfaces I and II reduces to $x = a$. This is the equation of the edge P_7P_1 with the end points left off, since z cannot be equal to a. The length of OA_x is now equal to $a[1 + \sqrt{\sqrt{2} - 1}]$.

$\pi < \omega < 2\pi$. In this case surface I alone exists and OA_x is still given by (8). This surface gradually closes down on the face of the cube as ω approaches $\pi/2$.

15. Forms of different level surfaces. By "level surfaces" we mean surfaces from which the cube has the same apparent size.

A. $0 < \omega < \pi/2$. In this case the surface consists of twenty-six distinct parts which form one continuous surface completely surrounding the cube and which approaches a sphere as ω approaches 0.

B. $\omega = \pi/2$. Eight of these surfaces now reduce to mere points, and the remaining eighteen form a continuous surface completely surrounding the cube and touching it at its corners.

C. $\pi/2 < \omega < \pi$. Only eighteen surfaces exist and they form a surface continuous everywhere except at the corners of the cube. This is of the same form as case *B* except the corners are left out.

D. $\omega = \pi$. Twelve of the surfaces reduce to lines (the edges of the cube), and the remaining six form a surface continuous everywhere except at the corners of the cube and touching the cube along its twelve edges.

E. $\pi < \omega < 2\pi$. Only six surfaces exist; one opposite each face. They form a surface completely surrounding the cube except that each edge is a line of discontinuity and each corner is a point of discontinuity.

RESIDUES OF CERTAIN SUMS OF POWERS OF INTEGERS.

By T. M. PUTNAM, University of California.

In Vol. XXXI, pages 329–333 of the *Quarterly Journal of Mathematics*, GLAISHER showed that, if p be an odd prime, and

$$H_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{(p-1)^n},$$

then, when n is not a multiple of $p-1$, $H_n \equiv 0 \bmod p$; and when n is a multiple of $p-1$, $H_n \equiv -1 \bmod p$. By using Fermat's Theorem we have

$$1/a^n \equiv a^{p-n-1} \bmod p,$$

so that, putting $p - n - 1 = r$, we obtain

$$H_n \equiv 1 + 2^r + 3^r + \cdots + (p-1)^r \bmod p.$$

Hence the sum of the rth powers of the first $p-1$ integers is congruent to -1, or zero, according as r is, or is not, a multiple of $p-1$.

These results of Glaisher, as well as some others, may be obtained in an elementary way by starting from the identity,[1]

$$p^{r+1} - p = {}_{r+1}C_1 S_r + {}_{r+1}C_2 S_{r-1} + \cdots + {}_{r+1}C_r S_1,$$

where $S_r = 1 + 2^r + 3^r + \cdots + (p-1)^r$.

By taking, successively, $r = 1, 2, 3, \cdots p-2$, we obtain

$$S_r \equiv 0 \bmod p.$$

If $r = p-1$, $a^r \equiv a^{p-1} \equiv 1 \bmod p$; hence $S_{p-1} \equiv p - 1 \equiv -1 \bmod p$.

Furthermore since $S_{r'} \equiv S_r \bmod p$ whenever $r' \equiv r \bmod p-1$, the residues of this series are known for every power r.

Since $a^{2k} \equiv (p-a)^{2k} \bmod p$,

$$S_{2k} \equiv 1^{2k} + 2^{2k} + 3^{2k} + \cdots + 3^{2k} + 2^{2k} + 1^{2k};$$

hence denoting by T_r the sum

$$1^r + 2^r + 3^r + \cdots + \left(\frac{p-1}{2}\right)^r$$

we obtain

$$2T_{2k} \equiv S_{2k} \bmod p.$$

Therefore, when $2k$ is not a multiple of $p-1$, $T_{2k} \equiv 0 \bmod p$; and when $2k$ is a multiple of $p-1$

$$T_{2k} \equiv -\tfrac{1}{2} \bmod p.$$

The determination of the residues of T_{2k-1} may also be made to depend upon

[1] C. Smith, *Treatise on Algebra*, 5th edition, p. 404.

the original identity. If the latter be written in terms of the T's it takes the form

$$\left(\frac{p+1}{2}\right)^{r+1} - \left(\frac{p+1}{2}\right) = {}_{r+1}C_1 T_r + {}_{r+1}C_2 T_{r-1} + \cdots + {}_{r+1}C_r T_1$$

or

$${}_{r+1}C_1 T_1 + {}_{r+1}C_2 T_2 + \cdots + {}_{r+1}C_r T_r \equiv \frac{1}{2^{r+1}} - \frac{1}{2} \bmod p.$$

By letting $r = 1, 2, 3, \cdots$, successively, the residues of T_1, T_2, T_3, etc., may be computed. It will be found as shown above that for any even value of r the residue of T_r is zero. The first seven odd values of r give the following results.

$$T_1 \equiv -\frac{1}{2^3}, \qquad T_3 \equiv \frac{1}{2^6}, \qquad T_5 \equiv -\frac{1}{2^7}, \qquad T_7 \equiv \frac{17}{2^{11}},$$

$$T_9 \equiv -\frac{31}{2^{11}}, \qquad T_{11} \equiv \frac{691}{2^{14}}, \qquad T_{13} \equiv -\frac{5461}{2^{15}}, \qquad \text{each taken mod } p.$$

These results hold for any prime p.

From a formula given by Glaisher in the *Quarterly Journal*, volume XXXII, page 279, a general expression for these residues may be derived. The formula is

$$\frac{1}{2^{2i+1}} + \frac{1}{4^{2i+1}} + \cdots + \frac{1}{(p-1)^{2i+1}} \equiv (-1)^{h-i}(2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \bmod p,$$

where $2h = p - 1$ and B_{h-i} is the Bernoulli number of rank $h - i$.

Remembering that $1/a^{2i+1} \equiv a^{p-2i-2} \bmod p$, and taking out $1/2^{2i+1}$ from the left member, we get

$$\frac{1}{2^{2i+1}}\left\{1 + 2^{p-2i-2} + 3^{p-2i-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2i-2}\right\}$$

$$\equiv (-1)^{h-i} \cdot (2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \bmod p.$$

Letting $h - i = k$, or $p - 2i - 2 = 2k - 1$, and $2i + 1 = p - 2k$, we get

$$T_{2k-1} \equiv (-1)^k \cdot 2^{p-2k}(2^{2k} - 1) \cdot \frac{B_k}{2k} \equiv (-1)^k \frac{2^{2k} - 1}{2^{2k}} \cdot \frac{B_k}{k} \bmod p.$$

This formula holds for $k = 1, 2, \cdots, \frac{1}{2}(p-1)$, but for $k = \frac{1}{2}(p-1)$ the denominator of B_k is divisible by p, so that it should be written

$$T_{p-2} \equiv (-1)^{\frac{1}{2}(p-1)} \cdot \frac{2^{p-1} - 1}{2^{p-1}p} \cdot \frac{pB_{\frac{1}{2}(p-1)}}{\frac{1}{2}(p-1)}$$

$$\equiv (-1)^{\frac{1}{2}(p+1)} \cdot 2 \cdot \frac{2^{p-1} - 1}{p} \cdot pB_{\frac{1}{2}(p-1)} \bmod p.$$

But $pB_{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \bmod p$.[1] Hence

$$T_{p-2} \equiv -2 \cdot \frac{2^{p-1} - 1}{p} \bmod p.$$

It follows from the above formulas that for no value of k is T_{2k-1} congruent to zero for every odd prime p, though there are, in general, special primes for each value of $k > 3$ for which T_{2k-1} will be congruent to zero.

The residue of T_{p-2} is seen to depend upon the residue of $(2^{p-1} - 1)/p$. The vanishing of this residue has been shown by Wieferich to be a necessary condition in order that Fermat's equation $x^p + y^p = z^p$ shall be solvable in integers prime to p. However, as stated on page 9 of *L'Intermédiaire des Mathématiciens*, January, 1914, this residue does vanish for $p = 1{,}093$, though for no other primes less than 2,000.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Plane and Solid Geometry. By WALTER BURTON FORD, Junior Professor of Mathematics in the University of Michigan, and CHARLES AMMERMAN, of the Wm. McKinley High School, St. Louis. Edited by EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1913. ix + 321 + xxxiii pages. $1.25. Plane and Solid in separate volumes $0.80 each.

Another interesting text-book of the series edited by Earle Raymond Hedrick, of the University of Missouri, has made its appearance; an attractive volume in a neat brown dress. The work, as we would expect, under Professor Hedrick's editorship, departs from the class of text-books written by scientists not in touch with life. The Euclidean division into books has been abandoned for the modern division into chapters.

The pupil is brought into touch with the subject matter of geometry in the introduction, covering 33 pages, which is divided into three parts: (I) Drawing simple figures, (II) the principal ideas used in geometry, (III) statements for reference. This division of the book contains 111 exercises. Squared paper is used to estimate areas and construct designs.

The arrangement of the subject matter of Plane Geometry in the first five chapters follows that of the usual modern standard text. Logical and original thinking are sure to be developed through the "judicious fusion of theoretical and applied work," here presented. There are approximately 750 exercises in these five chapters. Dr. Frank M. McMurry said, in a paper on "How to Study" given at the Minnesota State Teachers' Association last fall: "Do we in school know how to arrange the work so the pupil will learn to select? Teachers object to neglecting any part of the text. They dwell upon all parts with equal

[1] *Quarterly Journal*, Vol. XXXII, p. 273.

emphasis. Yet in life we have to select, and the exercise of choice is of first importance." The text in question helps the teacher to "exercise choice" very admirably. Different kinds of type are used to indicate the relative importance of theorems. Enough problems are given so that the exceptional pupil may have the stimulus he needs; the medium pupil a fair number; and the slow, plodding pupil enough to make the subject interesting and significant, yet not too difficult. The Appendix to Plane Geometry treats of maxima and minima.

The next three chapters—VI–VIII—are devoted to Solid Geometry. They are made attractive to look at by phantom half-tones. The reviewer would venture the criticism that these "phantom" half-tone engravings, while perhaps making the figures more realistic to the pupil, defeat one of the aims of solid geometry, that is, to develop the imagination. Many of the proofs of the theorems in this division of the book are given in outline only. There are 250 exercises in the Solid Geometry.

A decided innovation in this text is the introduction of three sets of tables. Table I gives the ratios of the sides of right triangles and chords and arcs of a unit circle. This indicates, of course, that the fundamental trigonometric ratios are introduced under proportion and similarity. All starred exercises indicate the use of these ratios. To quote from Article 112, "The relations between chords, arcs, and central angles of the same circle appear vividly in connection with rotations, ... Taking the radius as 1 unit, the lengths of the chords corresponding to various central angles for every degree from 0° to 90° are given in the table of chords." Table II gives powers and roots with an accompanying explanation for use. Table III gives a list of important numbers.

Very few historical notes are given. Such names as Pythagoras, Aristotle, Euclid and Bhaskara are mentioned in connection with famous theorems.

AGNES E. WELLS.

CENTRAL HIGH SCHOOL,
DULUTH, MINN.

A Text-Book of Mathematics and Mechanics, specially arranged for the use of students qualifying for science and technical examinations. By CHARLES A. A. CAPITO. Charles Griffin & Co., London; J. B. Lippincott Co., Philadelphia, 1913. xv + 398 pages.

As the title indicates this is a drill book for the use of persons preparing to take certain examinations; it is admirably adapted to this purpose and one can well believe that it grows out of a long and successful experience in coaching. The theory is presented concisely and, generally, with great clearness. Although extreme brevity characterizes the treatment throughout, space is devoted to the solution of numerous problems, most of which are taken from examination papers of recent date.

The author presumes a knowledge of elementary algebra and trigonometry, and begins with a section on plane analytic geometry, which is followed by sections on differential and integral calculus and mechanics, including hydromechanics

and pneumatics. The section on analytical geometry presents in 46 pages the common theorems relating to the straight line, the circle, and the other conic sections. This is followed by a set of 22 problems with a complete solution for each.

The section on differential calculus covers 59 pages, including 18 pages devoted to the solution of examples. This is brief indeed and quite unsatisfactory from the point of view of its future application in problems of mechanics and physics. In the introduction of the derivative, accuracy of statement is sacrificed to brevity; the process of passing to a limit is not carefully explained, the undetermined form 0/0 is scarcely more than mentioned and that in a confusing manner, and the equation

$$dy = f'(x)dx$$

is said to be a conventional form of the equation

$$\frac{dy}{dx} = f'(x),$$

introduced for the purpose of saving space. Aside from the Taylor and Maclaurin series the applications of the derivative are almost exclusively geometrical. Applications and interpretations of the derivative according to the method of rates, and the uses of the derivative in approximate computations by the method of differentials are practically excluded from consideration. Likewise, in the presentation of the integral calculus the process of integration is not treated as the limit of a process of summation but only as the reverse of the differentiation process. Thus those methods of the calculus which have proved most fruitful in the study of physical phenomena are set aside in favor of those which admit of brief and simple presentation with a minimum of logical difficulty.

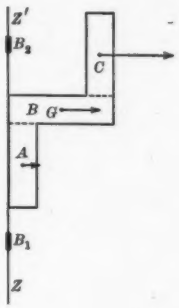
The presentation of Taylor's series is the one which was current in elementary text-books some twenty years ago when the function was assumed expressible in power series and the problem of determining the coefficients was solved by successive differentiations of the series, term by term.

The section on mechanics begins on page 170 and fills the remainder of the 398 pages of the book. The treatment here is in the main admirable, though always very brief. Liberal space (94 pages out of 228) is devoted to the solution of problems. Statics is disposed of in eight pages and kinematical discussions are omitted altogether. The space is therefore practically all devoted to the study of kinetics, with chapters on hydromechanics and pneumatics. The matter of dimension in terms of the fundamental quantities, mass, length and time, is kept well in the foreground. This feature contributes much to the value of the book.

In the words of the author this section "has been written with the intention of avoiding, as far as possible, the unscientific and erroneous expressions still employed by writers of the present day." Just what these expressions are does not appear very prominently, but the author has maintained a satisfactory standard of accuracy and rigor throughout this section. He objects to the use

of the term "centrifugal force," in which objection many of those interested would not concur. This term has been abused by some and confused by others but it has a legitimate place in the theory of mechanics. In defining the unit of time the author states that it is the second, which is 1/86164 of the time the earth takes to complete one revolution about its axis. The second is usually defined as 1/86400 of a mean solar day. It is undoubtedly important that attention should be called to the fact that, owing to its orbital motion, the earth does not complete a revolution about its axis in one mean solar day, but the definition substituted by Capito is open to the objection that it is accurate to five significant figures only, whereas the other is precise as a definition, and capable of expression in terms of other phenomena with an increasing degree of accuracy as the number and accuracy of observations increases.

The unit of mass is defined as ".001 of the mass (at 0° C.) of a standard piece of platinum, etc." The reason for mention of temperature in this connection is not obvious.



An error is committed in the derivation of expressions for the pressure exerted upon bearings when a body of irregular shape rotates about a fixed axis (see p. 326, formula 2). According to this formula the X component of the bearing stress due to the centrifugal action of a rotating mass would be found by assuming the mass to be concentrated at its mass center. That this is incorrect will appear by reference to the figure. The Z-shaped lamina ABC rotates about the shaft ZZ', each of the three sections exerting upon the shaft a force, as indicated by the arrows. Clearly the resultant of these three forces does not pass through G, the center of gravity of the whole mass.

Formally considered the error consists in setting

$$\Sigma mxz = Mr_g\gamma,$$

where r_g is the distance of the center of gravity from the Z axis and γ is the Z coördinate of the center of gravity. This equation is satisfied when the mass M has a plane of symmetry perpendicular to the Z axis. It may be satisfied even when there is no such plane of symmetry, but in general it is not satisfied.[1]

The book has been prepared with great care and attention to detail. Only one misprint was noticed and that was a misspelled word. Numerous figures are furnished and they are clear and helpful. Care has been exercised to maintain a high standard of logical consistency, though some conspicuous lapses have been made in this particular. Perhaps the most serious objection to the book is its failure to develop and utilize those methods of the calculus which have proved most fruitful in the physical sciences. However this criticism would undoubtedly apply with more or less force to most of the elementary calculus texts now in use. The book would not prove satisfactory as a text for use in a first course in any of the branches which it covers, but it would be very helpful in the field for which

[1] See "the theorem of parallel axes" in Routh's *Elementary Rigid Dynamics*, ed. 1905, p. 10.

it was intended, that is, as a review outline for persons preparing to take examinations in elementary mathematics and mechanics.

The plan and style of this book suggest very forcibly some of the advantages and disadvantages of an examination system. Definiteness, conciseness and a certain degree of precision are encouraged but there is danger of stimulating over-use or mis-use of the memory, and discouraging breadth and originality of view and interest in larger problems that cannot be handled adequately within the space of an examination period.

BURT L. NEWKIRK.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

Solutions of 408, 409, 410, 411, 412, 413, 414, and 415 have been received. A solution of 406 is desired.

418. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Form the algebraic equation whose roots are $a_1 = 2\cos(2\pi/15)$, $a_2 = 2\cos(4\pi/15)$, $a_3 = 2\cos(8\pi/15)$, and $a_4 = 2\cos(14\pi/15)$.

419. Proposed by GEORGE A. OSBORN, Massachusetts Institute of Technology.

Show that

$$(1^5 + 2^5 + 3^5 + \cdots + n^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 = 2(1 + 2 + 3 + \cdots + n)^4.$$

GEOMETRY.

Solutions of 437, 438, 439, 440, 443 have been received. Solutions of 427, 430, 432 and 433 are desired.

447. Proposed by HORACE OLSON, Chicago Illinois.

Given the edge of a regular tetrahedron, find the radius of the circumscribed sphere.

448. Proposed by S. W. REAVES, University of Oklahoma.

Through a given point P within a given angle to draw a line which shall form with the sides of the angle a triangle of a given area [Well's *New Plane Geometry*, (1909), p. 153].

CALCULUS.

Solutions of 352, 354, 355, 356, 357, 358, 359, 361, 362, and 366 have been received. Solutions of 332, 337, 339, 340, 342 are desired.

368. Proposed by PAUL CAPRON, Annapolis, Md.

Develop $\log_{10}(x/\sin x)$ and $\log_{10}(\tan x)/x$, each to three terms, as functions of $\log_{10}\sec x$ and show that if x is less than $7^\circ\ 15'$ then, to five decimals,

$$\log_{10} x = \log_{10}\sin x + 1/3\log_{10}\sec x = \log_{10}\tan x - 2/3\log_{10}\sec x.$$

369. Proposed by I. A. BARNETT, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

MECHANICS.

Solutions of 288, 289, 292, 293, and 294 have been received. Solutions of 268, 269, 274, 275, 277, 278, 279, 286, and 287 are desired.

296. Proposed by C. N. SCHMALL, New York City.

A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, ϕ the inclination of the plane to the horizon and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\phi}{l[1+\cos(\phi+\psi)]}.$$

297. Proposed by C. N. SCHMALL, New York City.

A shrapnel shell strikes the ground and then explodes, dispersing its fragments in all directions with a common velocity v. If A be the area of the ground covered by the fragments, and if the dimensions of the shell be neglected, show that $A = \pi v^4/g^2$.

NUMBER THEORY.

Solutions of 207, 210, 212, 216, and 218 have been received. Solutions of 189, 191, 192, 196, 200, 205, 208, 209, 211, 213, 214, and 215 are desired.

220. Proposed by E. T. BELL, Seattle, Washington.

Let $[m/n]$ denote the greatest integer that is not greater than m/n; and let the two sets,

$$\left[\frac{m}{m-1}\right]; \left[\frac{m+1}{m-2}\right]; \left[\frac{m+2}{m-3}\right]; \cdots; \left[\frac{2m-3}{2}\right],$$

$$\left[\frac{m-1}{m-1}\right]; \left[\frac{m}{m-2}\right]; \left[\frac{m+1}{m-3}\right]; \cdots; \left[\frac{2m-4}{2}\right],$$

be denoted by (A) and (B) respectively.

Prove that a necessary and sufficient condition that $2m-1$ be a prime number is that the excess of the number of even integers in (A) over the number of even integers in (B) shall be equal to the excess of the number of odd integers in (A) over the number of odd integers in (B).

221. Proposed by THOS. E. MASON, Bloomington, Indiana.

Find numbers x such that the sum of the divisors of x is a perfect square [Carmichael, *Theory of Numbers*, p. 17].

SOLUTIONS OF PROBLEMS.

ALGEBRA.

403. Proposed by C. N. SCHMALL, New York City.

A torpedo-boat 40 miles from shore strikes a rock, making a rent in her hull which admits water at the rate of 15 tons in 48 minutes. The ship's pumps can expel 12 tons in an hour. If 60 tons of water is sufficient to sink the boat, find the average rate of steaming so that it may reach the shore just as it is about to sink.

SOLUTION BY CHRISTIAN HORNUNG, Tiffin, O.

If x represents the rate per hour of steaming, then $40/x$ is the number of hours it takes to reach the shore, and in that time $15 \cdot 60 \cdot 40/48x$ tons of water will have entered the hull, and $12 \cdot 40/x$ tons will have been expelled. Hence $(15 \cdot 60/48) \cdot (40/x) - (12/1) \cdot (40/x) = 60$, and $x = 4\frac{1}{2}$.

Also solved by EMMA GIBSON, F. M. MORGAN, HORACE OLSON, CLIFFORD N. MILLS, WALTER C. ELLS, and J. W. CLAWSON.

404. Proposed by V. M. SPUNAR, Chicago, Illinois.

Show that

$$(a+b)(a+b-1)\cdots(a+b-n+1) = a(a-1)(a-2)\cdots(a-n+1)$$
$$+\binom{n}{1}a(a-1)(a-2)\cdots(a-n+2)b+\binom{n}{2}a(a-1)(a-2)$$
$$\cdots(a-n+3)b(b-1)+\cdots+b(b-1)(b-2)\cdots(b-n+1).$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We have

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots;$$

also

$$(1+x)^b = 1 + bx + \frac{b(b-1)}{2!}x^2 + \frac{b(b-1)(b-2)}{3!}x^3 + \cdots.$$

Multiply and equate the coefficients of x^n. Then

$$\binom{a+b}{n} = \binom{a}{n} + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n-2}\binom{b}{2} + \cdots + \binom{b}{n}.$$

If we multiply both members of this equation by n! we obtain the desired result.

Also solved by ELIJAH SWIFT, who proved the proposition by induction.

405. Proposed by E. J. MOULTON, Northwestern University.

Given the alternating series

$$S = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

(*a*) Let S_n be the sum of the first n terms of the series. Show that in order to make the difference $S - S_n$ numerically less than $1/2k$ (k a positive integer) it is necessary and sufficient to make $n = k$; hence S_{500} differs from S by less than .001. (*b*) Let S_n' be the sum of S_n and $1/2$ the $(n+1)$th term of the series. Show that the difference $S - S_n'$ is numerically less than $1/2n(n+1)$; hence S_{22}' differs from S by less than .001.

SOLUTION BY THE PROPOSER.

Denote the numerical value of the remainder after n terms of the series by $|R_n|$.

(*a*) If we compare the two series

$$|R_{n-1}| = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots$$

$$= \frac{1}{n(n+1)} + \frac{1}{(n+2)(n+3)} + \cdots,$$

$$|R_n| = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots$$

with the series

$$\frac{1}{2n} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+4} \cdots\right]$$

$$= \frac{1}{n(n+2)} + \frac{1}{(n+2)(n+4)} + \cdots,$$

we have at once $|R_{n-1}| > 1/2n > |R_n|$, from which the statements follow.

(*b*) From the preceding inequality, we see that

$$\frac{1}{2n} > |R_n| > \frac{1}{2(n+1)};$$

and hence, subtracting $1/2(n+1)$, we have

$$\frac{1}{2n(n+1)} > |R_n| - \frac{1}{2(n+1)} > 0.$$

Letting S be the value of the series, we see

$$S - S_n' = (-1)^n\left[|R_n| - \frac{1}{2(n+1)}\right]$$

and the statements (*b*) follow.

REMARK. Perhaps the chief interest of the problem comes from a comparison with a conclusion obtainable immediately from a well-known general theorem on alternating series (Osgood, *Introduction to Infinite Series*, p. 13), namely, that, for our series, to make $|R_n| < .001$ it is *suf-*

ficient to take $n = 1000$; we see that a much smaller number, 500, is also sufficient (and necessary). Part (*b*) of the problem was originally suggested to the proposer by the fact that when the values of $S_1, S_2, S_3, \cdots$ are plotted on a line, two successive points lie on opposite sides of S and about equally distant from S. The midpoint between S_n and S_{n+1} is S_n'. The value S_n' as an approximation to S may also be obtained as follows: We may write

$$S = S_n + (-1)^n\left[\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right]$$

$$= S_n + \frac{(-1)^n}{n+1}\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots\right]$$

$$= S_n + \frac{(-1)^n}{n+1} - (-1)^n\left[\frac{1}{(n+2)(n+3)} + \frac{1}{(n+4)(n+5)} + \cdots\right].$$

Adding the last two equations and dividing by 2, we have

$$S = S_n' + (-1)^n\left[\frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+3)(n+4)(n+5)} + \cdots\right].$$

If we compare this with the second equation we find

$$|S - S_n'| < \frac{1}{n+3} \cdot |S - S_n| \quad \text{or} \quad |S - S_n'| < \frac{1}{2n(n+3)},$$

which is a slightly better test than that given in the problem. In particular it follows that $|S - S_{21}'| < .001$.

407. Proposed by E. B. ESCOTT, University of Michigan.

In computing the values of the natural logarithms of 2, 3, and 5 by the following formulas:

$$\log 2 = 2(7P + 5Q + 3R),$$

$$\log 3 = 2(11P + 8Q + 5R),$$

$$\log 5 = 2(16P + 12Q + 7R),$$

where P, Q, and R are numbers which were computed by infinite series (G. Chrystal, Algebra, Part II, chapt. 28), it is found, on comparing the results with the known values of these logarithms to 15 decimals, that there are the following errors: -2533, -4052, and 6080, respectively. Find the errors in P, Q and R.

SOLUTION BY S. A. JOFFE, New York City.

Denoting the computed values of P, Q and R by the same capital letters with primes, and the errors of computation by the corresponding small letters, we will have: $P' = P + p$; $Q' = Q + q$; $R' = R + r$.

Since $\log 2 = 2(7P + 5Q + 3R) = 2[7(P' - p) + 5(Q' - q) + 3(R' - r)]$, and the computed value $\log' 2 = 2(7P' + 5Q' + 3R')$, we see that the error in $\log' 2$, when compared with the known value of $\log 2$, is $\log' 2 - \log 2$, or

$$2(7p + 5q + 3r) = -2533; \tag{1}$$

similarly:

$$2(11p + 8q + 5r) = -4052, \tag{2}$$

$$2(16p + 12q + 7r) = 6080. \tag{3}$$

This system of simultaneous equations may be solved as follows:

Multiplying (1), (2), and (3) by 4, -1, and -1 respectively, we eliminate q and r, and obtain $2p = -4 \times 2533 + 4052 - 6080$, or $p = -6080$.

Multiplying (1), (2), and (3) by -3, -1, and 2 respectively, we eliminate r and p, and obtain $2q = 3 \times 2533 + 4052 + 2 \times 6080$, or $q = 11{,}905\frac{1}{2}$.

Multiplying (1), (2), and (3) by -4, 4, and -1, respectively, we eliminate p and q, and obtain $2r = 4 \times 2533 - 4 \times 4052 - 6080$, or $r = -6078$.

Hence the errors in computing P, Q, and R are respectively -6080; 11905.5 and -6078.

Excellent solutions were also received from WALTER C. EELLS, J. W. CLAWSON, and the PROPOSER.

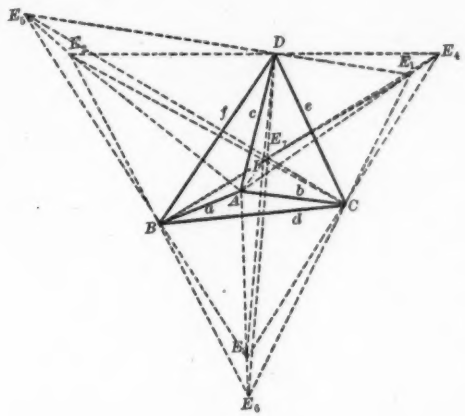
GEOMETRY.

417. Proposed by R. P. BAKER, University of Iowa.

Enumerate the points in which the twelve dihedral bisector planes of a tetrahedron meet, find their multiplicity and account for the 220 points which 12 planes in general determine.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Call the planes bisecting the dihedral angles internally by the letters a, b, c, d, e, f; the planes perpendicular to these which bisect the dihedral angles externally a', b', c', d', e', f'.



Six of these planes pass through the vertex A. Six planes in general determine 20 points. Now a, b, c intersect in the straight line joining A to the center of the inscribed sphere and the opposite ex-center; a', b, c' intersect in a line through A and two of the centers of escribed spheres; a, b', c' in a line joining A to another pair of ex-centers; a', b', c in another such line. These 20 points then reduce to a vertex, taken 16 times, and 4 straight lines passing through that vertex. Since there are four vertices, we can thus account for 80 of the 220 points.

Six of the 12 planes pass through the center of the inscribed sphere. Six planes in general determine 20 points. However a, b, c intersect in the straight line AI; and similarly the three planes passing through each of the other three vertices determine 3 lines. This eliminates 4 of the 20 points from consideration,

since these four lines have already been considered in the paragraph above. Thus the planes intersect three by three at I, taken 16 times. Similarly each of the seven ex-centers is determined 16 times. For instance a', b', f', c, d, e' intersect at E_6. These six planes determine the point E_6 16 times and four lines AE_6, BE_6, CE_6, DE_6, already considered above. This gives us then 8 points each repeated 16 times or 128 more of the 220 points.

Twelve points remain to be located. Now e intersects a and a' at a point on AB equidistant from two faces of the tetrahedron. And e' intersects a and a' at a second point on AB equidistant from the same two faces. Similarly there are two points on each of the six edges in which three planes intersect. This gives the remaining 12 points, each determined once.

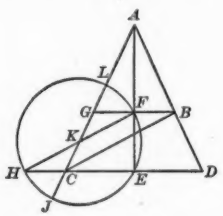
The 220 points, then, are four vertices, each 16 times; an in-center and 7 ex-centers each 16 times; 2 points on each of the 6 edges; and four straight lines from each of the four vertices to the in-center and ex-centers, these centers lying in pairs on these lines, 16 lines in all. Total 204 points and 16 lines. The figure shows these points and lines, except the 12 points, 2 on each edge.

434. Proposed by CLIFFORD N. MILLS, Bloomington, Ind.

ABC is any triangle with sides, a, b, c. Prove by purely geometrical methods that the area $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

SOLUTION BY E. E. WHITFORD, New York City.

Lay off $AG = AB$, draw GB, draw $CD \| GB$ meeting AB prolonged at D. Through the middle point F of GB draw AE meeting CD in E. Draw FH through K, the mid-point of GC, meeting CE prolonged at H. With K as center and KF as radius describe circle. AE is $\perp$ to GB and CD, $KF \|$ to CB and $HF = CB$. H and E are on the circumference.



$$\triangle ABC = \triangle ADC - \triangle BDC = CE \cdot AE - CE \cdot EF = CE \cdot AF.$$

By similar $\triangle$'s, $CE : AE = GF : AF$. Hence $CE \cdot AF = AE \cdot GF = AE \cdot HC$,

$$\triangle ABC = \sqrt{CE \cdot AF}\sqrt{CE \cdot AF} = \sqrt{CE \cdot AF}\sqrt{AE \cdot HC},$$

$$\triangle ABC = \sqrt{CE \cdot AF \cdot AE \cdot HC} = \sqrt{(AF \cdot AE) \cdot (CE \cdot HC)},$$

$$\triangle ABC = \sqrt{(AJ \cdot AL) \cdot (CL \cdot CJ)} = \sqrt{s(s-c)(s-a)(s-b)}.$$

For $AJ = AK + KJ = \frac{1}{2}(AC + AG) + KF = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}a = s$, $AL = AJ - JL = s - a$, $CL = GJ = AJ - AG = s - c$, $CJ = AJ - AC = s - b$.

This solution is based on Legendre.

Also solved by J. L. RILEY.

435. Proposed by R. M. MATHEWS, Riverside, Calif.

From a fixed point A perpendiculars are dropped to the tangents drawn to a circle whose center is O. Prove that the locus of the feet of the perpendiculars is a limaçon.

SOLUTION BY WALTER C. EELLS, U. S. Naval Academy.

Let a be the radius of the circle, and let distance OA be c. Take x-axis through O and A and y-axis through $O \perp OA$. Then the equation of the circle is $x^2 + y^2 = a^2$, and that of all tangents to it with the slope m is

$$y = mx \pm a\sqrt{1 + m^2}. \tag{1}$$

The equation of all lines through $A \perp$ to the system of lines given by (1) is

$$y = -\frac{1}{m}(x - c). \tag{2}$$

Transforming to new origin at A by equations $x = x' + c$, $y = y'$ and dropping primes, equations (1) and (2) become

$$y = m(x + c) \pm a\sqrt{1 + m^2}, \tag{3}$$

$$m = -\frac{1}{y}x. \tag{4}$$

Eliminating the parameter m by substituting (4) in (3), clearing of fractions, transposing and squaring, we have the well-known equation

$$(x^2 + y^2 + cx)^2 = a^2(x^2 + y^2),$$

representing the three standard forms of the limaçon, according as $c <, =, > a$ i. e., according as A is within, on, or without the circle.

Also solved by J. W. CLAWSON, S. W. REAVES, ELIJAH SWIFT, A. M. HARDING, DANIEL KRETH, C. N. SCHMALL, HORACE OLSON, and the PROPOSER.

CALCULUS.

Additional solutions were received too late for the June issue from J. L. RILEY for 335, and from A. M. HARDING and J. W. CLAWSON for 345.

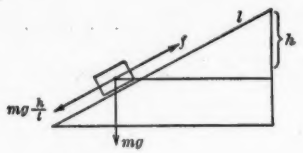
346. Proposed by C. N. SCHMALL, New York City.

Give the height of an inclined plane, to determine its length so that a given force acting on a given mass parallel to the plane may draw it up in the shortest time.

SOLUTION BY B. L. LIBBY, Ann Arbor, Mich.

Using the notations indicated in the figure, we have the resultant force acting upwards

$$f - mg \cdot \frac{h}{l}.$$



The acceleration required is therefore given by

$$a = \frac{fl - mgh}{ml}$$

Hence $dl/dt = (fl - mgh/ml)t$, and $l = (fl - mgh/2ml)t^2$, from which

$$t^2 = 2ml^2/(fl - mgh).$$

Differentiating, we have

$$t\frac{dt}{dl} = \frac{mfl^2 - 2m^2ghl}{(fl - mgh)^2}.$$

The usual test for a minimum for t gives

$$l = \frac{2mgh}{f} \quad \text{and} \quad t = 2m\sqrt{\frac{2gh}{f}}.$$

Also solved by HORACE OLSON and P. PEÑALVER.

347. Proposed by R. D. CARMICHAEL, Indiana Univ.

Show that the differential equation

$$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5} - 45\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4} + 40\left(\frac{d^3y}{dx^3}\right)^3 = 0$$

remains unchanged when the variables x and y undergo any projective transformation.

SOLUTION BY A. M. HARDING, University of Ark.

Any projective transformation can be reduced to a succession of integral transformations of the form

$$x = aX + bY + c, \qquad y = a_1X + b_1Y + c_1, \tag{1}$$

combined with the particular transformation

$$x = \frac{1}{X}, \qquad y = \frac{Y}{X}.$$

Hence it will be sufficient to show that the given equation remains invariant under each of these transformations. From (1) we find

$$y' = \frac{a_1 + b_1Y'}{a + bY'}, \quad y'' = \frac{(ab_1 - a_1b)Y''}{(a + bY')^3}, \quad y''' = \frac{(ab_1 - a_1b)[(a + bY')Y''' - 3b(Y'')^2]}{(a + bY')^5},$$

$$y^{iv} = \frac{(ab_1 - a_1b)[(a + bY')^2Y^{iv} - 10b(a + bY')Y''Y''' + 15b^2(Y'')^3]}{(a + bY')^7},$$

$$y^{v} = \frac{(ab_1 - a_1b)[(a + bY')^3Y^{v} - 15b(a + bY')^2Y''Y^{iv} - 10b(a+bY')^2(Y''')^2 + [105(a + bY')b^2(Y'')^2Y''' - 105b^3(Y'')^3]}{(a + bY')^9}.$$

From (2) we find

$$y' = Y - XY', \quad y'' = X^3Y'', \quad y''' = -X^4(3Y'' + XY'''),$$

$$y^{iv} = X^5[12Y'' + 8XY''' + X^2Y^{iv}]$$

$$y^{v} = -X^6[60Y'' + 60XY''' + 15X^2Y^{iv} + X^3Y^{v}].$$

It can be easily shown by substitution that each transformation leaves the given equation invariant.

Also solved by J. W. Clawson and Geo. W. Hartwell.

349. Proposed by C. N. Schmall, New York City.

If $y = a \cos (\log x) + b \sin (\log x)$, eliminate the constants a and b and obtain the equation

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + y = 0.$$

Solution by C. C. Steck, New Hampshire State College.

By differentiating the equation $y = a \cos (\log x) + b \sin (\log x)$, we obtain

$$\frac{xdy}{dx} = -a \sin (\log x) + b \cos (\log x).$$

Differentiating this and multiplying the resulting equation by x, we get

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} = -a \cos (\log x) - b \sin (\log x) = -y.$$

From the last two equations readily follows the desired result

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + y = 0.$$

Also solved by M. E. Graber, J. B. Smith, J. W. Clawson, P. Peñalver, C. Hornung, Elmer Schuyler, A. M. Harding, W. W. Beman, A. L. McCarty, H. L. Slobin, Clifford N. Mills, Francis Rust, I. A. Barnett, F. C. Reisler, G. W. Hartwell, S. W. Reaves, Richard Morris, Albert R. Nauer, Barnem Libby, and Walter C. Eells.

351. Proposed by C. N. Schmall, New York City.

In the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ are given e the eccentricity, and the angle φ which the normal at any point P (on the curve) makes with the major axis. If R is the radius of curvature at P prove that

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{\frac{3}{2}}}.$$

SOLUTION BY GEO. W. HARTWELL, Hamline University, St. Paul, Minn.

For the given ellipse

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \tag{1}$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}. \tag{2}$$

Also

$$\frac{dy}{dx} = \cot\varphi. \tag{3}$$

Making these substitutions in the usual formula for radius of curvature,

$$R = \frac{(1+\cot^2\varphi)^{3/2}}{\frac{b^4}{a^2y^3}} = \frac{a^2y^3}{b^4\sin^3\varphi}. \tag{4}$$

From (1) and (3)

$$b^2x = -a^2y\cot\varphi. \tag{5}$$

Substituting this value of x in the equation of the ellipse, we have

$$a^4y^2\cot^2\varphi + b^2a^2y^2 = a^2b^4.$$

Hence,

$$y^2 = \frac{b^4}{b^2+a^2\cot^2\varphi} = \frac{b^4\sin^2\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi} = \frac{b^4\sin^2\varphi}{a^2-(a^2-b^2)\sin^2\varphi}. \tag{6}$$

But $a^2 - b^2 = a^2e^2$. Hence

$$y^3 = \frac{b^6\sin^3\varphi}{a^3(1-e^2\sin^2\varphi)^{3/2}}$$

and

$$R = \frac{b^2}{a(1-e^2\sin^2\varphi)^{3/2}}.$$

But $b^2 = a^2(1-e^2)$. Hence,

$$R = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}.$$

Also solved by RICHARD MORRIS, C. C. STECK, F. M. MORGAN, J. W. CLAWSON, S. W. REAVES, and J. G. GONZALES.

MECHANICS.

A solution of 273 was received from J. W. CLAWSON too late for the June issue.

271. Proposed by B. F. FINKEL, Springfield, Mo.

A hollow spherical shell is filled with a frictionless fluid and rolls down a rough inclined plane. After rolling t seconds the fluid suddenly solidifies. Determine the subsequent motion of the spherical shell.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let m be the mass of contained fluid; m' the mass of the shell alone; α the angle of inclination of the plane; a the radius of sphere; x the distance the sphere

has rolled from rest after a time T; θ the angle turned through by the sphere after time T; I the moment of inertia of the rotating mass; and T' the time that has elapsed after solidification.

Using the formula for torque, we have

$$(m + m')ga \sin \alpha - (m + m')\ddot{x}a = I\ddot{\theta}.$$

But $x = a\theta$. Hence, $\ddot{x} = a\ddot{\theta}$. Hence, $(m + m')ga^2 \sin \alpha = (m + m')a^2\ddot{x} + I\ddot{x}$, from which we find

$$\ddot{x} = \frac{(m + m')ga^2 \sin \alpha}{(m + m')a^2 + I}.$$

Now, for the first t seconds, $I = 2/3 \cdot m'a^2$, since the fluid does not rotate. (The shell is assumed to be of so small a thickness that its square may be neglected in finding I.) Hence

$$\ddot{x} = \frac{(m + m')gs \text{ in } \alpha}{(m + m') + 2/3 \cdot m'}, \quad \dot{x} = \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot t, \quad x = \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot \frac{t^2}{2}.$$

After the first t seconds, $I = 2/3 \cdot m'a^2 + 2/5 \cdot ma^2$.

Hence,

$$\ddot{x} = \frac{(m + m')g \sin \alpha}{7/5 \cdot m + 5/3 \cdot m'}, \qquad \dot{x} = \frac{15(m + m')g \sin \alpha}{21\, m + 25m'} T' + \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot t,$$

and

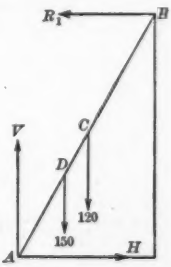
$$x = \frac{15(m + m')g \sin \alpha}{21\, m + 25\, m'} \frac{T'^2}{2} + \frac{3(m + m')g \sin \alpha}{3m + 5m'} T' \cdot t + \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot \frac{t^2}{2}.$$

285. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A ladder 25 ft. long, weighing 120 lbs., leans against a vertical wall. Its foot is prevented from slipping on the plane by a peg driven into the ground 7 feet from the wall. If a man weighing 150 lbs. is one-third the way up the ladder, what is the reaction on the peg, the ground, and the wall?

SOLUTION BY RICHARD MORRIS, Rutgers College.

Denote the reaction against the vertical wall by R_1, that against the peg by H and that against the ground by V, the weight of the ladder being concentrated at its mid-point.



Taking moments about A, we get

$$24\, R_1 = \frac{7}{3} \cdot 150 + \frac{7}{2} \cdot 120 = 770.$$

Hence $R_1 = 770/24 = 32\,{}^1/_{12}$ lbs. $R_1 = H$, since horizontal forces are equal. Also $V = 120 + 150 = 270$, resolving vertically.

Also solved by S. W. REAVES, J. B. SMITH, CIFFORD N. MILLS, A. M. HARDING, and J. W. CLAWSON.

NUMBER THEORY.

204. Proposed by E. T. BELL, New York City.

Show that a necessary and sufficient condition that $6n + 1$ be a prime number is that no one of the quantities $(3m - r)/(2r + 1)$ for $r = 1, 2, 3, \cdots n - 1$ be an integer. Similarly for $6n - 1$, the quantities being $(3n - r)/(2r - 1)$ for $r = 2, 3, 4, \cdots n$.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Suppose first that the condition be not satisfied. Then we have for some $r < n$, $(3n - r)/(2r + 1) =$ an integer, say α, which is equivalent to the equation $6n + 1 = (2r + 1)2\alpha + 2r + 1$. Hence, $6n + 1$ has the factor $2r + 1$ and is not prime, so that the condition is necessary.

Again suppose that $6n + 1$ is not prime. We shall see that the given condition is not satisfied. In this case the odd number $6n + 1$ has two odd factors, which we will call $2r + 1$ and $2\alpha + 1$. Each must be as large as 5 since 3 is evidently not a factor of $6n + 1$. If $2\alpha + 1 \geqq 5$, $2r + 1 \leqq (6n + 1)/5 = n + (n + 1)/5$. But this is less than $2n - 1$ if $n \geqq 2$, and if $n = 1$, $6n + 1$ is prime. Hence $r \leqq n - 1$. Reversing the algebraic work above we infer from $6n + 1 = (2r + 1)(2\alpha + 1)$ the equation $(3n - r)/(2r + 1) = \alpha$. Hence the condition is sufficient.

The proof of the second half of the theorem is similar mutatis mutandis.

206. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the sum of the sixth powers of two integers cannot be the square of an integer.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

We are required to show that the equation $a^6 + b^6 = c^2$, where a, b, c are integers, is impossible. We may suppose that a, b, and c are prime to each other, as any common factor may be divided out. Now all integral solutions of the equation $x^2 + y^2 = z^2$, where x, y, z are prime to each other are given by the formulas $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$, where m and n are integers prime to each other, and one is even and one odd. We must, then, have $a^3 = 2mn$, $b^3 = m^2 - n^2$. Suppose that m is even and n odd. Since $2mn$ is a cube, and $2m$ and n are prime to each other, each must be a cube, and $2m$ must contain the factor 8, and m the factor 4. Call $2m = 8\alpha^3$, $n = \beta^3$. Substituting these values in the formula for b^3, we have $b^3 = 16\alpha^6 - \beta^6 = (4\alpha^3 - \beta^3)(4\alpha^3 + \beta^3)$. Since $2m$ and n have no common factor, $4\alpha^3$ and β^3 have none, and $4\alpha^3 - \beta^3$ and $4\alpha^3 + \beta^3$ must also be prime to each other. Hence each is a perfect cube, since their product is, and we have $4\alpha^3 + \beta^3 = q^3$, $4\alpha^3 - \beta^3 = p^3$.

But from these two equations follows at once by addition $8\alpha^3 = p^3 + q^3$ or $r^3 = p^3 + q^3$, which is impossible. If n is even and m is odd the same method applies.

This method is general and may be applied to prove that if a, b, c, and n are positive integers (A) $a^{2n} + b^{2n} = c^2$ is possible only if there exist positive integers p, q, and r such that (B) $p^n + q^n = r^n$. If we assume the truth of Fermat's

theorem—namely that (B) is impossible if $n > 2$ — we have shown the impossibility of (A) for every value of n except $n = 1$, or $n = 2$. For $n = 1$ (A) is of course possible; its impossibility for $n = 2$ may be shown by this same method. See Bachmann, *Niedere Zahlentheorie*, vol. II, p. 453.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTIONS.

13. What are the most important of the newer problems that confront teachers of high school mathematics, and why are they important?

REPLY.

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics?

I. REMARKS BY EDITH LONG, Lincoln, Nebraska.

That there has been and still is an unrest among thoughtful high school teachers of mathematics is manifest. This problem certainly exists: How are we to adapt the courses in algebra and geometry to the experiences of high school students, in order that these may contribute to their broader development in both a cultural and a practical way? On the one hand we have the student who is preparing for the university and on the other hand the student whose school life ends with the high school or who, going on to the university, will not continue the subject of mathematics. The one who would solve this problem must have these two classes in view, and the solution for the one must be the solution for the other since it is only in the largest schools that distinction can be made by having separate classes, even if it were advisable. It is for the high school teacher to work out a simple course which will serve the purpose of college preparation as well as the one now followed and which at the same time will give to the boy or the girl who does not go on a clear and simple notion of the nature and possibilities of mathematics as a whole. It ought to be possible for pupils to leave the high school with other impressions than that this great subject is embodied in two narrow and distinct divisions called algebra, where one works exercises, and geometry, where one demonstrates theorems. Some impressions of tangible and present value should be made to replace the vague feeling of subtle virtue, not to be understood now, but which in future years it is hoped will show its value and repay for long and weary hours spent.

There is no conflict between these two aims. A course that will satisfy the latter purpose will serve the former far better than those we now have. But before such a course can be formulated and put into effect the barriers of prejudice must be broken down. The tandem plan of algebra, geometry, trigonometry, analytical geometry, and calculus, running through the high school, and the first two years of college, should give way to a new course which will unify these sub-

jects. The simple parts with concrete basis should be given in the high school and from this on in one continuous course the subject should be developed in the college. It is, indeed, strange that this plan has been reversed and that much of the clear, simple mathematical intuitions are postponed to the college while much abstract work is required of the ninth-grade, thirteen-year-old child; for example, in connection with the theory of exponents and radicals and the binomial theorem, which apart from the claims of formal discipline, can be of no use to any one until they are applied in the higher mathematics. But any child of average brain power can understand and enjoy every phase of either algebra or geometry which is developed from measurement and drawing; and in this are to be included the elementary notions of analytical geometry and trigonometry which, to avoid paralyzing names, should come under the head of graphic work.

He who runs may read the handwriting on the wall. Either we must adapt the course to the student or it will be swept entirely from the curriculum in the frenzied demand for humanized and vocational courses of study. Viewed purely from the standpoint of preparation for college, the great slaughter of students at the end of the first semester points to the weakness of the present plan. How could it be otherwise under an arrangement which is so contrary to the principles of pedagogy and child-psychology? First comes algebra, written in a language entirely strange to the child, with rules to be memorized and used in drill work for rapidity and accuracy. Now algebra is held in the schools under the supposition that it gives reasoning power, but such memory work does not stimulate thought power. And of what use is accuracy if there is no thought back of it? It is accuracy of thought primarily and not merely accuracy of execution that should be sought in mathematics. If drill in mechanical accuracy and rapidity are the aim, a course in business arithmetic would answer the purpose better.

After a year of drill, when the student is beginning to acquire some efficiency in this line, another extreme kind of mental activity is introduced. Geometry, as it is taught, demands intensive deductive reasoning in place of rule and drill. This is continued for a year when the child is suddenly switched back to his drill work for another half year. Then it is all dropped until a year and half later the student is called upon, not to continue his algebra and geometry in the way in which he has been taught, but to apply them in ways in which he has not been taught. It is not strange that so many fail in the first year of college mathematics.

From the standpoint of the course finished in the high school, and this is really the more important, there can be no doubt that readjustment must take place if the student is to reap the best results. To be successful this readjusted course must be built upon the experiences of the child in connection with the measurement of lines, areas and solids, first brought into use by drawings, so as to appeal to the eye, and then by expression in algebraic symbols. Thought conveyed to the brain through the eye is lasting. The picture appeals and lingers. There is little of algebra which cannot be based on some picture and none that should be required of ninth-grade students. As far as possible the picture thought

should be constantly seen in the algebraic sentence. No sentence without thought should be demanded in algebra as it is in other studies. Thought to the child means concrete sight thought. It is not only a child trait but a universal trait that the thought which the eye conveys is most effectively conceived. Data expressed in a list of figures make little impression, while a graph from the same list is convincing. A well-drawn cartoon may do more damage to a cause than a season of speeches. The time spent in drill may be lessened by one half if in the algebraic symbols is seen a shorthand writing of an English sentence, and at the same time the student will acquire a steady thoughtful rate of manipulation.

For example, why is the thermometer the best illustration of negative quantities? Because in the thermometer we have a perfect mechanical graph. It has the scale line, an arbitrarily chosen zero, an arbitrary unit of measure, and a real little point running up and down the scale, adding both positive and negative quantities. Children delight in working with it. The picture of this is the graph and the student can always draw this picture. But the picture has broader possibilities. It can be made to illustrate quantity of every kind that admits of the positive and negative idea.

But it is not for the sake of training the student to be more expert in the solution of exercises that I make the plea for the concrete treatment of algebra. It is for the sake of giving as a basis of the entire development something tangible, something definite upon which the student can build the work for himself. Having in hand material which he understands, he should be able to organize his work and develop the subject with some independence. It should not be necessary for the teacher to take the lead every time a new phase of the subject comes up. With a clear notion that factoring has its counterpart in finding the dimensions of rectangles whose areas are given, and previously having studied the nature of the rectangle, the student should by drawing find his own way to factoring, first arithmetic numbers, and then simple quadratic expressions. The teacher should guide, not develop, the work. To always carefully develop a subject before assigning new work may do more harm than to allow a class sometimes to flounder, for in the latter case there are always some who will find their way out with great strength, while in the former case every member of the class is crippled. With material in hand and lessons carefully assigned there will be no floundering, not even when the student passes from algebra to geometry. A student who has developed his own algebra from geometric figures scarcely realizes that there is any change in subject. He has already learned to take the initiative in organizing his work and he is not handicapped. As the geometry progresses he can work out the simpler notions of trigonometric ratios and thus widen the field of application to real things about him.

In this reconstruction we need the coöperation of the university teacher. While it is true that many school superintendents declare that they pay no attention to what the university demands, nevertheless, the present arrangement of the mathematical course in the high school was originally made through the influence of college entrance requirements and nothing would facilitate the readjustment

now so much as a liberal attitude on the part of the colleges and universities in accepting the unified course for admission. There is no line along which the child-study investigators could better direct their attention than to placing mathematics on a sounder pedagogical basis; for there is no subject which causes such waste of time and such humiliation through failure on the part of the child.

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

An article (in German) by Professor T. H. Gronwall of Princeton University on the summability of Laplace and Legendre series appears in the June number of *Mathematische Annalen.*

Professor C. J. Keyser discourses upon "The Study of Mathematics" in his inimitable style in the *Columbia University Quarterly* for June.

Mr. W. W. Kustermann has been appointed to an instructorship in mathematics in the University of Michigan, to succeed Mr. E. B. Escott who resigned recently.

Professor R. C. Archibald has been added to the editorial board of the *Bulletin* of the American Mathematical Society.

At Purdue University Mr. Ralph B. Stone has been promoted to an assistant professorship of mathematics; Dr. Thomas E. Mason, Mr. O. W. Albert and Mr. Charles K. Robbins have been appointed to instructorships; Professor William Marshall has been given leave of absence for a year of study in France.

The June issue of *School Science and Mathematics* contains a paper by Professor W. B. Ford of the University of Michigan on "The Future of Geometry" read before the spring meeting of the Michigan Schoolmaster's Club; and a paper by Miss I. E. Holroyd entitled "Mathematics in the Education of Girls" read before the Mathematical Club of the Kansas State Agricultural College.

Mr. W. A. Reinert, of the Oak Park, Ill., high school, has been appointed to an instructorship in mathematics at the Michigan Agricultural College, Lansing, Mich. He is a bachelor of the University of Wisconsin and has been taking graduate work at the University of Chicago.

Professor David Eugene Smith of Teachers College, Columbia University, has leave of absence for the next academic year. He attended the Napier tercentenary at Edinburgh where he made an address, and also the Roger Bacon celebration in Oxford, where he read a paper on "Roger Bacon as a mathematician."

The Mathematics Teacher for June prints an interesting address by Dean F. C. Ferry of Williams College on "Mathematics: The Subject and the Teacher."

Mr. M. G. Gaba, who has just taken the doctor's degree at the University of Chicago, has been appointed to an instructorship in mathematics in the Carnegie Institute of Technology at Pittsburg, Pa.

Professor F. B. Wiley, who has been on leave of absence from Denison University, Granville, Ohio, for the purpose of graduate study, took the doctor's degree in mathematics at the University of Chicago at the convocation held in August.

Science, May 8, 1914, contains a short note on the first meeting and the organization of the "Committeee of One Hundred on Scientific Research," which was recently appointed by the American Association for the Advancement of Science. Ten members of this committe are connected with bureaus of the federal government. The educational institutions which are represented on this committee by at least three men each are as follows: Harvard (12), Chicago (9), Columbia (7), Yale (5), Johns Hopkins (5), Carnegie Institution (5), Massachusetts Institute of Technology (4), Cornell (3), Illinois (3), and Stanford (3). Charles W. Eliot, president emeritus of Harvard University, is chairman of the committee, which will hold its next meeting at Philadelphia on December 28, 1914. The committee aims to consider broad questions relating to research "under the government, in the universities and in other institutions."

A Heft of the *Encyklopädie der Mathematischen Wissenschaften*, devoted to elementary geometry, appeared in June of the present year. It discusses many questions which are of interest to the teachers of elementary geometry in the high schools. The parts of this encyclopedia which have been published contain about ten thousand pages. Several additional parts are in press.

The May, 1914, number of the *Revista de la Sociedad Matemática Española* contains a brief account of the first congress of mathematical philosophy which was held at Paris during last April.

An interesting article on the teaching of mathematics in Japan appeared in the May number of the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht.*

The mathematics section of the California High School Teachers Association met at the University of California July 2 and 3, 1914. Among the papers presented were the following: "How shall the isolated teacher of mathematics keep up his interest?" by Professor D. N. Lehner, University of California; "Mathematics and life in the intermediate school," by Miss Thirmuthis Brookman, special instructor in the University of California summer session; "The

university and the secondary schools," by Dr. Henry W. Stager, of Fresno Junior College, and chairman of the section. A fuller report of this meeting will appear in a later issue of the MONTHLY.

At the University of Minnesota W. H. Bussey has been promoted to an associate professorship in mathematics, and H. L. Slobin and W. F. Holman have been promoted to assistant professorships.

The Macmillan Company of London has recently published "A first book of geometry" by J. V. H. Coates as one of the series entitled "First Books of Science." It contains only 142 pages, and sells for 50 cents.

Isis, the new journal devoted to the general history of science, will in the future publish an *edition de luxe* on thin paper for the convenience of bibliographers and librarians.

In connection with the commemoration exercises of the seventh centenary of Roger Bacon's birth a volume of essays on various aspects of his activity has been published by the Oxford University Press. Professor David Eugene Smith is one of the contributors. His paper is entitled "The place of Roger Bacon in the history of mathematics." Bacon's works on optics, mirrors, physics, alchemy and chemistry, and medicine are also treated in special essays. The volume is edited by A. G. Little; the other contributors are L. Baur, F. Picavet, Cardinal Gasquet, S. A. Hirsch, Eilhard Wiedemann, S. Vogl, J. Wurmschmidt, Pierre Duhem, M. M. P. Muir, H. W. L. Hime, E. Withington, and J. E. Sandys.

Columbia University will commemorate Bacon's birth by appropriate exercises at the opening of the college year this fall. A pageant is planned as part of the program. The Research Club of the University of Michigan devoted the April meeting to addresses in honor of Bacon. Professors Dow, Tatlock, Lloyd, and Guthe contributed the papers. Professor Guthe's paper on "Roger Bacon as a scientist" appeared in the August issue of *The Open Court Magazine*.

Volney H. Wells, A.B. Olivet College, graduate student 1912–1914 at the University of Michigan, has been appointed instructor in mathematics at the University of Michigan.

Professor Elijah Swift, of Princeton University, has been appointed head of the department of mathematics at the University of Vermont.

Miss Susan Rose Benedict, fellow in mathematics at the University of Michigan in 1913–1914, has been promoted to an associate professorship of mathematics at Smith College. Miss Benedict received the Ph.D. in June at the University of Michigan. Her thesis is entitled, "A comparative study of the early treatises which introduced into Europe the Hindu art of reckoning."

Paul E. Hemke and L. E. Williams have been appointed to instructorships in mathematics at the Georgia School of Technology, Atlanta, Ga. They were both graduate students at the University of Chicago.

Professor E. R. Hedrick read a paper at the June meeting of the Society for the Promotion of Engineering Education held at Princeton University, on the subject "The Calculus without Symbols." This paper is published in full by the Society in its volume of proceedings. Special interest attaches to such a discussion in view of the current tendency to introduce the elements of the calculus into secondary instruction.

The first part of the *Encyclopédie des Sciences Mathématiques* appeared in 1904. It was then expected that the entire work could be issued in about 50 parts, each containing about 150 pages. Although more than 30 parts have now been published, it appears likely that more than 50 additional parts will be needed to complete the work. Hence we seem to be farther from the end than they thought they were at the beginning of the publication. More than 20 parts are now in press. It should, however, not be assumed that all of these will appear soon, as some parts have been in press for several years.

Those particularly concerned with the teaching of secondary algebra will be interested in the experiment by Professor E. L. Thorndike of Teachers College, Columbia University, reported in *The Mathematics Teacher* for March. A list of problems in algebra was graded as to difficulty by two hundred teachers of mathematics. There is a satisfactory agreement in the results, notwithstanding some striking differences in estimating the relative difficulty of certain problems. Each of the following was rated all the way from the easiest to the hardest in point of difficulty: (1) "If $a = 6$ and $b = 3$, what does $\sqrt{a}\sqrt{2b}$ equal?" (2) "Find the average midnight temperature for the week in which the daily midnight temperatures were 15, 3, 0, -7, -9, 6, and 17 degrees." (3) "At what time between 6 and 6:30 o'clock are the hands of a watch at right angles to each other?"

The Cambridge University Press publishes "A Shorter Geometry" by Godfrey and Siddons which is written in the spirit of the recommendations of the British Board of Education concerning the teaching of geometry. The course is organized in three stages: (1) Introductory work concerned with the fundamental notions and not primarily designed to give facility in using instruments; (2) Discovery by experiment and intuition of the fundamental facts about angles at a point, parallels, angles of a triangle and polygon, congruent triangles; the accurate use of instruments and the elementary ideas of logical argument; (3) Deductive development of the propositions which can be derived from those propositions discovered by experiment and intuition in the second stage. The proofs of the propositions which were enunciated in the second stage are grouped in an appendix which may be postponed at pleasure or "till they are needed for

examination purposes". It is the expressed belief of the Board that the deductive proofs of the propositions thus postponed are more difficult and less persuasive than those that follow, that difficulties of sequence center here very largely, and that more rapid treatment of these is conducive to better results in subsequent work.

In extension of the summary of recent work on the southerly deviation of a falling body given in the MONTHLY of December, 1913, it may be said that a paper entitled "Deviation of Falling Bodies" by Professor F. R. Moulton of the University of Chicago appears in the June number of the *Annals of Mathematics*. Professor Moulton concludes that when the approximations introduced in the process are properly analyzed, the deviation of a body freely falling a small distance near the earth's surface is found to be equator-ward for all latitudes between 0° and $\pm$ 90°.

The International Commission on the Teaching of Mathematics met at Paris April 1 to 4 inclusive with an enrolment of 160, half of whom were from France. It was unfortunate that the time of the meeting made impossible a representative attendance from England or America, only six from the former and one from the latter being present. The chief interest centered about the sectional meetings where the subjects of discussion were the reports on (*a*) the results of the introduction of the elementary notions of the calculus into secondary instruction, and (*b*) the mathematical instruction of engineering students. *L'Enseignement* for May 15 gives a very full account of this meeting, including brief summaries of what are said to be admirable reports on the two topics mentioned above. Later notice will be given in these columns of the publication of the complete reports.

The Commission will hold its sessions next year in Munich, at which time it is expected that the reports of the subcommissions of all countries will be complete. It will be remembered that the American reports were all completed for presentation at the time of the meeting of the Commission in Cambridge in 1912, and that, as is probably known to all teachers of mathematics, these may be obtained free on application to the U. S. Bureau of Education, Washington, D. C.

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REMARKS ON KLEIN'S "FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY."

By RAYMOND CLARE ARCHIBALD, Brown University.

No teacher of geometry in a high school should be ignorant of the contents of this little book which appeared in German nearly twenty years ago, but which is also available in an English translation.[1] We here find: (1) the exact statement of the proof of the necessary and sufficient conditions for constructions with ruler and compasses; (2) the proofs of the impossibility of solution, by means of ruler and compasses, of the famous problems: trisection of an angle, duplication of the cube, squaring the circle; (3) full discussion of Gauss's results concerning regular polygons constructible with ruler and compasses; (4) general considerations on algebraic constructions, the integraph, and the geometric construction of π, and other facts of interest in elementary geometry.

While reading this book with a class in geometry for teachers, I had occasion to criticize and elaborate certain parts. In somewhat condensed form, my notes in this connection are given in the following pages.

Gaussian Polygons.—Up to the time of Gauss, no one suspected that it was possible to construct, with ruler and compasses, regular polygons other than those the number of whose sides could be expressed in one of the forms: 2^n, $2^n \cdot 3$, $2^n \cdot 5$, $2^n \cdot 15$. All of these were known to the Greeks. But Gauss proved as early as 1801[2] that whenever a prime number p could be expressed in the form $2^{2^m} + 1$, the construction of a regular polygon with p sides was possible by Euclidean methods. It was then apparent that regular polygons not included in the Euclidean series, namely of 17, 257, 65537, . . . sides, could be constructed

[1] F. KLEIN, (*a*) *Vorträge über ausgewählte Fragen der elementar Geometrie*, ausgearbeitet von F. Tägert. Leipzig, Teubner, 1895, 66 pp. +2 Tafeln. (*b*) *Famous Problems of Elementary Geometry* . . . authorized translation of F. Klein's Vorträge . . . , by Wooster Woodruff Beman and David Eugene Smith. Boston, Ginn, 1897, ix+80 pp. (*c*) *Leçons sur certaines questions de géométrie élémentaire* . . . rédaction Française autorisée par l'auteur, par J. Griess. Paris, Vuibert, 1896, 100 pp. There was also an Italian edition published at Turin in 1896, and a Russian edition which appeared in Kasan in 1898.

[2] *Disquisitiones arithmeticæ*, Lipsiæ, 1801, *Werke*, Bd. 1, 2, abdruck 1870, p. 462; French ed. *Recherches Arithmétiques*, Paris, 1807, p. 488; Ger. ed. by Maser, Berlin, 1889, p. 447.